INDEPENDENT SETS IN REGULAR GRAPHS

BY

M. ROSENFELD*

ABSTRACT

Lower and upper bounds for the maximal number of independent vertices in a regular graph are obtained, it is shown that the bounds are best possible. Some properties of regular graphs concerning the property $\mathscr{H}$ defined below are investigated.

Introduction. In this paper we are interested in independent sets in regular graphs. In §2 we give bounds for the maximal number of independent vertices in a regular graph G. (G will always denote a graph without loops and multiple edges). It is shown that these bounds are best possible. It seems true that each value between the bounds is obtainable. In §3 we define the property $\mathscr{H}$ for graphs: we say that a graph $G \in \mathscr{H}$ if every vertex of G belongs to a maximal independent set of vertices in G. In some cases conditions are given under which $G \in \mathscr{H}$. In §3 we define a class of graphs called homogeneous for which it seems to be interesting to investigate their properties and structure.

1. Definitions and Notations

A graph G will be called regular of degree m if every vertex is incident with exactly m edges. We shall denote such a graph by $G(n,m)$ where n is the number of vertices in G. It is evident that such a graph exists iff $n > m$ and $n \cdot m \equiv 0 (\text{mod}\, 2)$.

$\alpha(M)$ will denote the number of elements of the finite set M. $\bar{G}$ will denote the complementary graph of G. The components of G are its maximal connected subgraphs.

A set $R = \{a_1 \cdots a_k\} \subseteq S$ (S the set of vertices of G) will be called a representing system of the edges of G if every edge of G is incident with at least one vertex from R. $\mu(G)$ will denote the minimal number of vertices representing the edges of G. $\bar{\mu}(G)$ will denote the maximal number of independent vertices in G. $\nu(G)$ denotes the number of edges in G.

$M \subseteq S$. $[M]$ will denote the subgraph spanned by M. $M, N \subseteq S$, a MN-edge is an edge whose one endpoint is in M and the other in N. $[M,N]$ will denote the subgraph whose vertices are $M \cup N$ and the MN-edges contained in G. $a, b \in S$, $(a,b) \in G$ denotes that a and b are incident in G. C_n will denote the complete graph with n vertices.

Received January 20, 1965.

* This paper is to be a part of the author's Ph. D. thesis written under the supervision of Prof. B. Grünbaum at the Hebrew University of Jerusalem.

$[\beta]^*$ will denote the smallest integer not less than β.

2. THEOREM I. *Let $G = G(n, m)$ be a regular graph of degree m with n vertices. Denote by $\bar{\mu}(G)$ the maximal number of independent vertices in G then:*

$$\text{a) } \bar{\mu}(G) \leqq \min \left\{ \left[\frac{n}{2}\right], n - m \right\}.$$

$$\text{b) } \bar{\mu}(G) \geqq \begin{cases} k+2 & n = k(m+1) + m; \ m > 2 \\ 3 & G = G(2n+1, 2n-m); \ m < n; \frac{5m}{2} > 2n+1 \\ \left[\dfrac{n}{m+1}\right]^* & \text{in all other cases.} \end{cases}$$

These bounds are best possible in the sense that for each pair of integers n, m such that $m < n$ and $n \cdot m \equiv 0 \pmod 2$ they are obtainable.

Proof. Let S be the set of vertices of G. Let $A \subseteq S$ be an independent set of vertices in G and $\alpha(A) = \bar{\mu}(G)$. Since A is a maximal independent set of vertices, each vertex of $S - A$ is neighboring with at least one vertex of A. The number of different edges having one endpoint in A is $\bar{\mu}(G) \cdot m$ hence:

1) $\bar{\mu}(G) \cdot m \geqq n - \bar{\mu}(G) \Rightarrow \bar{\mu}(G) \geqq \left[\dfrac{n}{m+1}\right]^*$ ($\bar{\mu}(G)$ is an integer!)

2) $\nu(G) = \dfrac{n \cdot m}{2} \Rightarrow \dfrac{n \cdot m}{2} \geqq \bar{\mu}(G) \cdot m \Rightarrow \left[\dfrac{n}{2}\right] \geqq \bar{\mu}(G).$

Let $a \in A$. The m different edges whose one endpoint is the vertex a have their second endpoint in $S - A$. Therefore

3) $\alpha(S - A) = n - \bar{\mu}(G) \geqq m \Rightarrow \bar{\mu}(G) \leqq n - m.$

To show b) we need two Lemmas:

LEMMA I. *Let* $G = G(K(m+1) + m; m)$, $m > 2 \Rightarrow \bar{\mu}(G) > K + 1$

$$\left(\text{i.e. } \bar{\mu}(G) \geqq \left[\frac{n}{m+1}\right]^* + 1\right).$$

Proof. (Observe that m must be even otherwise the graph does not exist). To prove the lemma we use induction on k.

For $k = 1$: Since $G = G(2m+1, m) \Rightarrow \bar{G} = G(2n+1, m)$ and since an independent set of k vertices in G form a complete k-subgraph in $\bar{G}$ it will suffice to show that $\bar{G}$ must contain a C_3.

Let $a \in S$ be any vertex in G. $a \to \{a_1 \cdots a_m\} = A$. If a is not contained in a C_3 A must be independent. $S = \{a\} \oplus A \oplus B$. $\alpha(B) = m$. This implies that each $a_i \in A$ is an endpoint of $m - 1$ edges whose second endpoint is in B.

$$v(G) = m^2 + \frac{m}{2} \Rightarrow v([B]) = m^2 + \frac{m}{2} - (m + (m-1)m) = \frac{m}{2}.$$

Since by our assumption $m > 2$ and m is even $v([B]) \geqq 2$. Let $b_i \to b_k$, $b_i' \to b_k'$ (it is possible that $b_i = b_i'$ but $b_k \neq b_k'$) b_i is neighboring with some $a_i \in A$ otherwise $b_i \oplus A$ would be an independent set with $m+1$ vertices which is impossible. If $a_i \nrightarrow b_k$ $a_i \to \{B - b_k\}$ $a_i \to b_i'$ and $a_i \to b_k'$ and $[a_i b_i' b_k'] = C_3$. If $a_i \to b_k$ then $[a_i b_i b_k] = C_3$.

Suppose $k_0 > 1$ is the smallest positive integer for which the lemma does not hold. Let us denote by G_0 the graph satisfying

$$(1) \qquad G_0 = G(k_0(m+1) + m_1 m) \quad \bar{\mu}(G_0) = k_0 + 1.$$

G_0 cannot be connected: if G_0 is connected by a theorem due to Brooks [1] G_0 is m-chromatic. Hence

$$G_0 = \sum_{i=1}^{m} \oplus A_i$$

where A_i is the set of vertices colored with the i-th color.

$$\max \alpha(A_i) \geqq \frac{k_0(m+1) + m}{m} > k_0 + 1$$

Since A_i is an independent set of vertices in G_0 this is a contradiction to our assumption (1). Hence G_0 must have at least two components:

$$(2) \qquad G_0 = G_1 \oplus G_2.$$

Let us consider the two possible cases:

i) $G_1 = G(k_1(m+1) + r_1, m)$ $G_2 = G(k_2(m+1) + r_2, m)$ $k_1 + k_2 = k_0$, $r_1 + r_2 = m$ $r_2, r_1 \neq 0$

ii) $G_1 = G(K_1(m+1) + m, m)$ $G_2 = G(K_2(m+1), m)$.

Case (i): $\bar{\mu}(G_0) = \bar{\mu}(G_1) + \bar{\mu}(G_2) \geqq K_1 + 1 + k_2 + 1 = K_0 + 2$ which contradicts the assumption (1) hence this decomposition is impossible.

Case (ii): By the induction hypothesis $(K_1 < K_0)\, \bar{\mu}(G_1) \geqq K_1 + 2$

$$\bar{\mu}(G_0) = \bar{\mu}(G_1) + \bar{\mu}(G_2) \geqq K_1 + 2 + K_2 = K_0 + 2.$$

This shows that $\bar{\mu}(G_0) = K_0 + 1$ is impossible and the proof of the lemma is completed.

Lemma II. *Let* $G = G(2n+1; m)$ *and suppose that* $\frac{5m}{2} > 2n + 1$ *then* G *contains a* C_3.

Proof. Suppose that G does not contain a C_3. Let $a_1 \to \{b_1, b_2 \cdots b_m\} = B$ $b_1 \to \{a_1 \cdots a_m\} = A$. If G does not contain a C_3, A and B are independent sets of vertices and $A \cap B = \emptyset$. Therefore $G = A \oplus B \oplus C$ where $C = \{c_1 \cdots c_r\}$ and by our assumption r in an odd integer and $r < \frac{m}{2}$. Let $c \in C$ and suppose that c has $r_1 \leqq r - 1$ neighboring vertices in C. Suppose furthermore that

$$c \to \{a_{i_1} \cdots a_{i_j} b_{k_1} \cdots b_k\} \qquad j + k = m - r_1 .$$

Without loss of generality we may suppose that $j \geqq k$. Since B is independent b_{k_1} has m neighboring vertices in $A \oplus C$. Now c has r_1 neighboring vertices in C; then if G does not contain a C_3 b_{k_1} has at most $r - r_1$ neighboring vertices in C. Therefore b_{k_1} has at least $m - r = r_1$ neighboring vertices in A. A contains $m - j$ vertices that are not incident with c.

$$j + k = m - r_1 \qquad j \geqq k \qquad m - j \leqq \frac{m + r_1}{2}$$

Since $\frac{m}{2} > r$, $m - r + r_1 > \frac{m}{2} + r_1 \geqq \frac{m + r_1}{2} \geqq m - j$; this means that b_k must have at least one neighboring vertex from $a_{i_1} \cdots a_{i_j}$ and G_0 would contain a C_3, a contradiction to our assumption. Therefore we conclude:

i) $c \to a_i \Rightarrow c \not\to \{B\}$.

ii) $c, c' \in C$, $c \to c'$, $c \to \{A\} \Rightarrow c' \to \{B\}$.

For if we had $c \to \{a_{i_1} \cdots a_{i_j}\} = A_c \subseteq A$, and $c' \to \{a'_{i_1} \cdots a'_{i_k}\} = A_{c'}, k = m - r_1 > \frac{m}{2}$, $j = m - r_1 > \frac{m}{2}$ $A_c \cap A_{c'} \neq \emptyset$ and G_0 would contain a C_3 (N_1, r_1 have the same meaning as in the preceding paragraph). Denote by $C_A = \{c_1 \cdots c'_j\} \subseteq C$ and $C_B = \{c'_1 \cdots c'_k\} \subseteq C$ those vertices of C having neighboring vertices in A or B respectively. Because of i) and ii) $C = C_A \oplus C_B$ and C_A and C_B are independent sets in G_0.

If G_0 does not contain a C_3 we conclude from the above discussion that G_0 contains only four types of edges:

(1) $C_A C_B$-edges (2) AC_A-edges (3) BC_B-edges (4) AB-edges.

Henceforth to complete our proof it will suffice to show that $v[B, C_B] \neq v[A, C_A]$. (This means that if G_0 does not contain a C_3 it cannot be regular).

Since r is odd and $j + k = r$ we may suppose that $k > j$. Let $r_i = d_{[C]}(c_i) \to$ we have $\sum_{i=1}^{j} r_i$ $C_A C_B$-edges in G_0, $j \cdot m - \sum_{i=1}^{i} r_i$ AC_A-edges and $k \cdot m - \sum_{i=1}^{i} r_i$ BC_B-edges. Since $k > j$ $k \cdot m - \sum r_i = v[B \cdot C_B] > j \cdot m - \sum_{i=1}^{i} r_i = v[A \cdot C_A]$

Q.E.D.

Lemmas I and II give the justification of the modification of the lower bound for $\bar{\mu}(G)$ given in (b). To complete the proof we will construct for each given admissible m, n a regular graph in which we obtain the bounds.

(a) $G(2n,m)\quad m\leqq n\quad \bar{\mu}(G)=n.$

$$A=\{a_1\cdots a_{..}\}\quad B=\{b_i\cdots b_n\}$$

$$a_i\to\{b_i,\ b_{i\dotplus 1}\cdots b_{i\dotplus m}\}\qquad i\dotplus m=\begin{cases} i+m & i+m\leqq n\\ i+m-n & i+m>n.\end{cases}$$

It is easily seen that this construction gives a regular bipartite graph with $\bar{\mu}(G)=n$. Observe that this graph does not contain a C_3 hence the complementary graph is a $G(2n,m')m'>n$ and $\bar{\mu}(G)=2$.

(b) $G(2n+1,m)\quad (m\leqq n)\quad \bar{\mu}(G)=n.$

Observe that in this case m must be even $m=2k$. To the graph constructed in (a) adjoin an additional vertex c. Omit the edges (a_ib_i) $1\leqq i\leqq k$ and add the edges (ca_i) and (cb_i). The resulting graph is easily seen to be a $G(2n+1,m)$ with $\bar{\mu}(G)=n$. Observe that if $k\geqq 2$ G will contain a C_3 but not a C_4.

(c) $G(n;m)\quad m>\dfrac{n}{2}\quad \bar{\mu}(G)=n-m.$

Let $G^*=C_{n-m}\oplus G(m\,;n-m-1)$.

$G(m;\,n-m-1)$ always exists: for $m>\dfrac{n}{2}$ implies that $m>n-m-1$, if m is odd n must be even and therefore $n-m-$ 1is even. Furthermore it is easily seen that G^* contains a C_{n-m} but not a C_{n-m+1}. The graph $\bar{G}^*=G(n,\,m)$ is the desired example. (a) (b) and (c) show that the upper bound stated in (a) is best possible for each admissible m, n.

(d) $G(k(m+1)+m;m)\quad \bar{\mu}(G)=k+2\quad m\geqq 4$ (See Lemma I).

For $k=1$ the graph $G_0(2m+1,m)=G(2m+1,m)$ where $G(2m+1,m)$ is the graph constructed in (b) has $(G_0)=3$. For $k>1$ the graph $G=C_{m+1}\oplus\cdots\oplus C_{m+1}\oplus G_0(2m+1,m)$ will have

$$\bar{\mu}(G)=k+2.$$

(e) $G(2n+1;m)\quad m=2r_0>n\quad 2n+1\geqq\dfrac{5(2n-m)}{2}\quad \bar{\mu}(G)=2.$

The condition $2n+1\geqq\dfrac{5(2n-m)}{2}$ is necessary because of Lemma II.

Denote the set of vertices by:

$$A=\{a_1,\cdots a_n\}\qquad B=\{b_1,\cdots b_n\}\ \text{and}\ c.$$

Edges: $c\to\{a_1\cdots a_r,\ b_1\cdots b_r\}\ m'=2rm'=2n-m$

$$a_i\to\{b_{r+1}\cdots b_n\}\qquad b_i\to\{a_{r+1}\cdots a_n)\qquad 1\leqq i\leqq r.$$

By our assumption $m'>n-r\Rightarrow n-r+1=m'-k\quad k\geqq 0.$

$$a_{r+i} \to \{a_j\}; \qquad b_{r+i} \to \{b_j\} \qquad 1 \leqq j \leqq r \qquad 1 \leqq i \leqq k.$$

This is possible since $n - r > k$ as will be shown in the sequel. It is easily seen that:

$$d(a_i) = d(b_i) = \begin{cases} 2r = m' & 1 \leqq i \leqq r+k \\ r & r+k+1 \leqq i \leqq n \end{cases} \qquad d(c) = 2r = m'.$$

Now $n - r - k = n - r - [m' - (n - r + 1)] = 2n + 1 - 2m' \geqq r$ by our assumption (which proves that $n - r > k$) hence we can construct from the two sets: $A' = \{a_{r+k+1}, \cdots, a_n\}$ and $B' = \{b_{r+k+1}, \cdots b_n\}$ a bipartite regular graph of degree r with A', B' independent. It is easy to see that the graph thus defined is a $G(2n+1, m')$ that does not contain a C_3. The complementary graph of this graph is the example looked for.

(f) $G(n,m)$ $\bar{\mu}(G) = \left[\dfrac{n}{m+1}\right]^*$ $n \neq k(m+1) + m$ $\dfrac{5m'}{2} \leqq n$ (if n is odd).

Let $n = h(m+1) + r$ $0 \leqq r < m$.

If $r = 0$ take $G = C_{m+1} \oplus \cdots \oplus C_{m+1}$ (h times).

If $r > 0$ but $m + 1 + r$ is even or $m + r + 1 \geqq \dfrac{5r}{2}$ take in the first case $G = C_{m+1} \oplus \cdots C_{m+1} \oplus G_a$ and in the second case:

$$G = C_{m+1} \oplus \cdots \oplus G_e$$

Where $G_a = G(m + r + 1, m)$ is the graph constructed in (a) ($\bar{\mu}(G_a) = 2$) and $G_e = G(m + r + 1, m)$ is the graph constructed in (e). One easily verifies the equality $\bar{\mu}(G) = \left[\dfrac{n}{m+1}\right]^*$. If $m + r + 1$ is odd but $m + r + 1 < \dfrac{5r}{2}$, m must be even.

Let $2m + r + 2 = 3k + j_0$ $0 \leqq j_0 \leqq 2$ ($2m + r + 2$ is even!). If $j_0 = 0$ k is even. Let $C^i = \{c_1^i \cdots c_k^i\}$ $1 \leqq i \leqq 3$ be a set of vertices. Join by an edge the following vertices:

$$c_i^1 \to \{c^2_{i+j(mod\,(k/2))}\} \quad c_l^1 \to \{c^3_{k-l+j(mod\,(k/2))}\} \quad c_l^2 \to \{c^3_{j+l}\}$$

$$1 \leqq i \leqq \frac{k}{2}; 0 \leqq j \leqq m - k \left(\leqq \frac{k}{2}\right); \frac{k}{2} + 1 \leqq l \leqq k; l \dotplus j = \begin{cases} l + j \text{ if } l + j \leqq k \\ l + s - 1 \text{ if } l + j = k + s \cdot s > 0. \end{cases}$$

Each C^i is a complete k-graph. These relations define a graph $G^* = G(3k, m)$ with $\bar{\mu}(G^*) = 3$. The graph $G = C_{m+1} \oplus \cdots \oplus G^*$ (C_{m+1} taken $h - 2$ times) is a $G(n,m)$ with $\bar{\mu}(G) = \left[\dfrac{n}{m+1}\right]^*$. If $i_0 \neq 0$ then:

$$2m + 2 + r = \begin{cases} 2(k + (k+1) & k+1 \text{ is even} \\ 2(k+1) + k & k \text{ is even.} \end{cases}$$

In the first case take three complete graphs: $C^1 = C^2 = C_k$; $C^3 = C_{k+1}$. In the second case we take $C^1 = C^2 = C_{k+1}$; $C^3 = C_k$. Add to the graph $C^1 \oplus C^2 \oplus C^3$ the following edges:

$$c_i^3 \to \{c^1_{i+j(mod\,(k+1)/2))}\} \qquad c^3_{k-i+1} \to \{c^2_{i+j(mod\,(k+1)/2))}\} \qquad c_s^1 \to \{c^2_{s+j}\}.$$

$$1 \leqq i \leqq \frac{k+1}{2} \qquad 0 \leqq j \leqq m-k-1 \qquad \frac{k+1}{2} < s \leqq k.$$

It is easily seen that these relations define a graph $G^* = G(3k+1, m)$ with $\bar{\mu}(G^*) = 3$. A similar construction can be carried out in the second case. In both cases the graph looked for is $G = C_{m+1} \oplus \cdots \oplus G^*$.

(g) $G(2n+1, 2n\text{-}m)$ $m \leqq n$ $\frac{5m}{2} > 2n+1$. $\bar{\mu}(G) = 3$

It is trivially seen that the complementary graph of the graph constructed in (b) is the desired example. This completes the proof of the theorem.

REMARKS. Since the complement of a maximal independent set of vertices in a graph is a minimal representing system and vice versa, our theorem can be applied to the estimation of $\mu(G)$ in regular graphs. P. Erdös and T. Gallai [2] have shown that:

$$n - \bar{\mu}(G) = \mu(G) \leqq \frac{2\nu(G)\pi(G)}{2\nu(G) + \pi(G)} \Rightarrow \bar{\mu}(G) \leqq \frac{n}{m+1}$$

which is the same bound obtained in (b). But in the case of regular graphs as was shown we can say more than in the general case. In [2] it is shown that the equality

$$\bar{\mu}(G) = \frac{n}{m+1}$$

holds if and only if G is the direct sum of complete graphs. In the case of regular graphs the equality

$$\bar{\mu}(G) = \left[\frac{n}{m+1}\right]^*$$

which can be obtained except for the two cases mentioned in (b) does not determine uniquely the graph in general. Furthermore, we can give a lower bound for $\mu(G)$: (only for regular graphs)

$$\mu(G) \geqq \max\left\{\left[\frac{n+1}{2}\right], m\right\}.$$

and the minimal value is obtainable for each n, m such that $n \cdot m \equiv 0 \pmod 2$.

2) Given a regular graph we can easily estimate $\varepsilon(G)$.

$$\varepsilon(G) = \bar{\mu}[I(G)]$$

where $I(G)$ is the interchange graph of G and if $G = G(n, m)$

then
$$I(G) = G(\tfrac{1}{2} n \cdot m,\ 2(m-1))$$
hence:
$$\left[\frac{n \cdot m}{4m-2}\right]^* \leqq \varepsilon(G) \leqq \min\left\{\left[\frac{n \cdot m}{4}\right],\ \frac{1}{2} n \cdot m - 2(m-1)\right\}.$$

One can easily modify these bounds using the fact that each vertex in $I(G)$ is contained in a C_m.

Lemma I is a sharpening of a theorem of Turán [4], for the case of C_3. Observe that it holds only for regular graphs having an odd number of vertices.

One can easily deduce from theorem I, that if $n/n-m > r$, $G(n,m)$ must contain a C_{r+1}. (This result can be obtained from Turán's theorem). A slight modification of this result can be obtained from lemma II:

If
$$G = G(u(m+1)+m, (u-1)(m+1)+m)$$
then G contains a C_{u+2}.

3. Definition A graph G will have the property $\mathscr{H}$ $(G \in \mathscr{H})$ if every vertex in G is contained in a maximal independent set of vertices with $\bar{\mu}(G)$ vertices.

In this section we shall investigate the property $\mathscr{H}$ in some extremal cases of regular graphs. For this we need few more definitions:

1) With each vertex of $G(n,m)$ we associate a $(m/2)$-tuple of integers ordered by increasing magnitude and defined as follows: with each two edges incident with the vertex in consideration associate the length of the shortest circuit containing them. If such a circuit does not exist the number associated will be $+\infty$. We denote by $\tau(a)$ the $(m/2)$-tuple associated with the vertex a, and call it the type of a. It is obvious that a necessary condition that there exists an automorphism of the graph that carries a to b is $\tau(a) = \tau(b)$.

2) A regular graph will be called homogeneous if all the vertices in each component have the same type. Examples of homogenous graphs are circuits, complete graphs and point symmetric regular graphs. A homogenous graph need not be point symmetric, see for example the graph constructed by B. Grünbaum in [3]

Lemma 3.1. *G is a graph.* $d(a) \leqq m \quad \forall a \in G \Rightarrow \bar{\mu}(G) \geqq \left[\dfrac{n}{m+1}\right]^*.$

Proof. We use induction on n. $(\pi(G) = n)$. For "small" n's the lemma is obvious. Let $a \in G$. $a \to \{b_1 \cdots b_{m'}\} m' \leqq m$. $\alpha(S - \{a\, b_1 \cdots b_{m'}\}) = n - (m'+1)$. The graph $G' = [S - \{a\, b_1 \cdots b_{m'}\}]$ satisfies the conditions of the lemma hence $\bar{\mu}(G') \geqq \left[\dfrac{n-(m'+1)}{m+1}\right]^* \geqq \left[\dfrac{n}{m+1}\right]^* - 1$ therefore a maximal independent set from G' together with a is an independent set A with $\alpha(A) \geqq \left[\dfrac{n}{m+1}\right]^*$.

Theorem 3.1. $G = G(n \cdot m)$. $\bar{\mu}(G) = \left[\dfrac{n}{m+1}\right]^* \Rightarrow G \in \mathscr{H}$.

Proof. The proof is a direct consequence of lemma 3.1. since we have shown that any arbitrary chosen vertex belongs to a maximal independent set.

THEOREM 3.2. $G = G(n,m)$ $\bar{\mu}(G) = \dfrac{n}{m+1} \Rightarrow G \in \mathscr{H}$ and G is uniquely determined.

Proof. P. Turán in [4] proved that in this case G is the direct sum of complete $m+1$-graphs, hence the theorem follows. We give here another proof of the theorem. We use induction on $k = \dfrac{n}{m+1}$. For $k=1$ the theorem is obvious since in this case $G = G_{m+1}$. Let $G = G(n,m)$ and $\bar{\mu}(G) = \dfrac{n}{m+1} = k > 1$ be given. If G is connected by a theorem of Brooks [1], G would be m-chromatic, $G = \sum_{i=1}^{\mu} \oplus A_i$, A_i is the set of vertices colored "i".

$$\max \alpha(a_i) \geqq \frac{n}{m} > \frac{n}{m+1}$$

But A_i is independent, in contradiction to the assumption $\bar{\mu}(G) = \dfrac{n}{m+1}$. Hence $G = G_1 \oplus G_2$ where $G_1 = G(n_1 m)$, $G_2 = G(n_2, m)$.

$$n_1 + n_2 = n \qquad \text{and} \qquad \bar{\mu}(G) = \bar{\mu}(G_1) + \bar{\mu}(G_2).$$

Suppose $\dfrac{n_1}{m+1}$ is not an integer $\Rightarrow \dfrac{n_2}{m+1}$ is not an integer by Theorem I:

$$\bar{\mu}(G) = \bar{\mu}(G_1) + \bar{\mu}(G_2) \geqq \left[\frac{n_1}{m+1}\right]^* + \left[\frac{n_2}{m+1}\right]^* \geqq \left[\frac{n_1+n_2}{m+1}\right] + 1 > \frac{n}{m+1}.$$

Hence: $\dfrac{n_1}{m+1} = k_1 < k$ $\quad \dfrac{n_2}{m+1} = k_2 < k$ and by the induction Hypothesis they are the direct sum of complete $m+1$-graphs. This means that G is point symmetric $\Rightarrow G \in \mathscr{H}$.

THEOREM 3.3. *Let $G = G(n,m)$; $m > \frac{1}{2}n$; $\bar{\mu}(G) = n - m$ then:*

1) $G \in \mathscr{H}$ if and only if $n - m/n$ and G is homogeneous. In this case G is uniquely determined and point symmetric.

2) $G \notin \mathscr{H}$ if b does not belong to a maximal independent set of vertices while a does then $\tau(b) < \tau(a)$. (The types are ordered lexiocographically).

Proof. 1) Suppose $G \in \mathscr{H}$. Let $A_1 = \{a_1, \cdots, a_{n-m}\}$ be a maximal independent set. Hence $a_i \rightarrow \{S - A_1\}$. $1 \leqq i \leqq n-m$. Therefore if $A_2 = \{a'_1 \cdots a'_{n-m}\}$ is a maximal independent set different from A_1 we must have $A_1 \cap A_2 = \emptyset$. Now if $G \in \mathscr{H}$ each $g \in G$ belongs to a uniquely determined maximal independent set of vertices, hence G is the direct sum of independent sets of vertices and each vertex is connected by an edge to all the vertices not belonging to the independent set including it. This means that $n - m/n$. Since the complementary graph of A is easily seen to be the direct sum of complete $n-m$ graphs, G is point symmetric and therefore homogeneous. It is then obvious that if $n-m \nmid n \Rightarrow G \notin \mathscr{H}$.

2) Let $a \in A$, A a maximal independent set $\Rightarrow a \to \{S-A\}$. Let B be the set of all vertices that are not joined by edge to b (including b) $\Rightarrow \alpha(B) = n-m$. Since each $a \in A$ is joined by an edge to $S-A$ and $b \in S-A$. $B \cap A = \emptyset$.

Hence: $G = C \oplus A \oplus B$.

Denote by $l(xay)$ the length of the shortest circuit containg the edges (ax) and (ay) (in the sequel it will be shown that $l(xay)$ is finite). Let us calculate $\tau(a)$ and $\tau(b)$.

Put $\{c_1 \cdots c_s\} = C \qquad s = 2m-n.$

$$c_i c_j \in C \quad \text{and} \quad (c_i c_j) \in G \qquad l(c_i a c_j) = l(c_i b c_j) = 3$$

$$(c_i c_j) \notin G \qquad l(c_i a c_j) = l(c_i b c_j) = 4.$$

$l(a_i b c_i) = 3$ this contributes $(n-m)(2m-n)$ times "3" to $\tau(b)$.

$l(a_i b a_j) = 4$ this contributes $\frac{1}{2}(n-m)(n-m-1)$ times "4" to $\tau(b)$.

Since b does not belong to a maximal independent set:

$$v[B] = r \geqq 1$$

Suppose therefore $(b_i b_j) \in G \Rightarrow \exists c', c'' \in C \wedge (b_i c'),\ (b_j c'') \notin G. \Rightarrow l(b_i a b_j) = 3$, $l(b_i a c') = 4$, $l(b_j a c'') = 4$.

Since $v[B] = r$ it is easily seen that we have $2r$ triangles of type $l(abc)$ more than of type $l(cab)$ while only r triangles of type $l(bab)$ more than of type $l(aba)$; this shows that in $\tau(b)$ we have r "3" more than in $\tau(a) \Rightarrow \tau(b) < \tau(a)$. This proves also that if G is homogenous we must have $n-m/n$ and $G \in \mathscr{H}$. This completes the proof of the theorem.

THEOREM 3.4. $G = G(n,m)$ $\bar{\mu}(G) = 3$. If a does not belong to a maximal independent set of vertices in G while b does then $\tau(a) < \tau(b)$.

Proof. Observe first that if $m < \frac{n}{2}$, $G \in \mathscr{H}$, hence we will suppose that $m \geqq \frac{n}{2}$.

Let $a \to \{x_1 \cdots x_m\} = X_a$, $a \nrightarrow \{y_1 \cdots y_{n-m-1}\} = Y_a$.

1) $G = \{a\} \oplus X_a \oplus Y_a$

2) a does not belong to a maximal independent set implies $Y_a = C_{n-m-1}$.

3) The number of different triangles containing a is $v[X_a]$.

$$v[X_a] = \frac{n \cdot m}{2} - \left\{ m + \frac{(n-m-1)(n-m-2)}{2} + (n-m-1)(2m-n+1) \right\}.$$

Let $b \to \{r_1 \cdots r_m\} = R_b \quad b \nrightarrow \{p_1 \cdots p_{n-m-1}\} = P_b$.

$$G = \{b\} \oplus R_b \oplus P_b.$$

Since b belongs to a maximal independent set $P_b \neq C_{n-m-1}$ the number of different triangles containing b is $v[R_b]$. It will therefore suffice to show that $v[R_b] < v[X_a]$. Suppose that in $[P_b]$ r edges are needed to complete the graph $[P_b]$, the two endpoints of such an edge are connected by an edge to vertices in R_b. Hence for each "missing" edge in P_b we have two "additional" PR-edges:

$$v[R_b] = \frac{n-m}{2} - \left\{m + \frac{(n-m-1)(n-m-2)}{2} - r + (n-m-1)(2m-n+1) + 2r\right\}$$

Since $r \geqq 1 \Rightarrow v[R_b] < v[X_a]$. This completes the proof.

REMARKS. 1) $G = G(n, m)$ $\bar{\mu}(G) = 3$ G is homogeneous $\Rightarrow G \in \mathscr{H}$.

2) In the general case we do not know when $G \in \mathscr{H}$; it is obvious that:

$G \in \mathscr{H} \Leftrightarrow \cap R_G = \emptyset$ (R_G runs over all the minimal representing systems in G)

but this is not a useful criterion.

REFERENCES

1. R. L. Brooks, *On coloring the nodes of a network.* Proc. Cambridge Philos. Soc., **37** (1941), 194–197.

2. P. Erdös and T. Gallai, *On the minimal number of vertices representing the edges of a graph* Magyar Tud. Akad. Mat. Kutató Int. Közl., **6** (1964), 181–202.

3. B. Grünbaum, *A problem in graph coloring.* (Unpublished).

4. P. Turán, *On the theory of graphs,* Coll. Mathematicum, **3** (1954), 10–30.

THE HEBREW UNIVERSITY OF JERUSALEM